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# Exact solution of a two-dimensional Ising model in a correlated random magnetic field

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**Abstract.** The Ising model in a random magnetic field $h$ is studied on a square lattice. $h$ has the same value along a row, but varies randomly from row to row. $h = \pm\infty$ with probability $p/2$ and is zero with probability $1-p$. The quenched free energy $F$ and specific heat $C$ are calculated exactly. We find $C \sim \ln p$ for small $p$ at the Onsager temperature $T_c$. It is shown that for $p \neq 0$, $F(p, T)$ is not analytic in $T$ at $T = T_c$, i.e. it has a Griffiths-like singularity.

## 1. Introduction

The behaviour of Ising spin systems in a random magnetic field (RMF) is a question still far from being settled. The phase transition is drastically altered in nature or completely suppressed. The values of critical exponents are largely changed and their determination is an outstanding problem. One of the most interesting questions is the lower critical dimensionality, $d_{lc}$, below which an infinitesimally small RMF prevents any kind of ordering. The Imry-Ma domain argument (Imry and Ma 1975) is believed to give the correct result for $d_{lc}$ in the case of systems with continuous symmetry ($d_{lc} = 4$). The same argument for Ising models (IM) predicts $d_{lc} = 2$. This prediction, however, has been the subject of controversy for many years. The number of publications, supporting $d_{lc} = 2$ (Imry and Ma 1975, Grinstein and Ma 1982, Villain 1982, Villain *et al* 1983, Imbrie 1984) and $d_{lc} = 3$ (Parisi and Soúrlas 1979, Binder *et al* 1981, Pytte *et al* 1982, Mukamel 1982, Niemi 1982) is of the same order. Most of the theoretical works aiming at the determination of $d_{lc}$ are either approximate, or use extrapolations from rigorous results obtained at non-physical dimensionalities. Recently, however, a strong mathematical argument has been given in favour of $d_{lc} = 2$ (Fisher *et al* 1984). Physical realisations of RMF are random uniaxial antiferromagnets as pointed out by Fishman and Aharony (1975). Although the well-pronounced effect of RMF on the phase transition in IM has been experimentally demonstrated (Wong *et al* 1982 and references therein) the question of $d_{lc}$ could not yet be resolved.

Information from exactly soluble models is rather scarce and even that is restricted to one dimension (Aeppli and Bruinsma 1983, Györgyi and Rujan 1984, Sütő and Zimanyi 1984). Using a special probability distribution for the magnetic field, Grinstein and Mukamel (1983) solved a 1D IM. They were able to calculate the largest linear

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size which a ferromagnetically ordered cluster (chain) of spins can attain. As a function of randomness the result for this length was consistent with $d_{lc} = 2$. Using the same probability distribution, Pelcovits and Mukamel (1983) solved the 1D $XY$ model, whereas Forgacs *et al* (1984) solved the 1D kinetic Ising model.

All the above mentioned works dealt with uncorrelated RMF, i.e. the field is an independent random variable at each site of the lattice. Correlated randomness has been studied in the case of random exchange. McCoy and Wu (1968) solved a 2D IM in which the nearest-neighbour couplings are the same between two subsequent rows of spins, but otherwise vary randomly from row to row. In higher dimensions expansions in $\varepsilon_d$ (the defect dimensionality) were used together with the renormalisation group (Dozegoutsev 1980, Boyanovsky and Cardy 1982). The question of lower and upper critical dimensionality in the case of correlated randomness was studied by Aharony *et al* (1982).

In the present paper (a short account of our work has been given earlier: see Forgacs *et al* (1985)) we study a 2D IM in the presence of correlated ($\varepsilon_d = 1$) RMF. Our model is defined by

$$\beta H = -K \sum_{ij} s_{ij}(s_{i,j+1} + s_{i+1,j}) - \sum_{ij} h_i s_{ij}. \tag{1.1}$$

Here $s_{ij} = \pm 1$, $i$ and $j$ denote the row and column indices of an $L \times M$ square lattice, respectively. $K = \beta J$, $\beta^{-1} = k_B T$, is the non-random exchange coupling ($T$ is the absolute temperature). The field $h_i$ has the same value within a row, but varies randomly from row to row. We choose $h_i = +\infty, -\infty$ or 0 with probabilities $p/2$, $p/2$ and $(1-p)$, respectively. For $M = 1$, this model reduces to the ones discussed before (Aeppli and Bruinsma 1983, Györgyi and Rujan 1984, Sütő and Zimanyi 1984). The model defined above could be called the magnetic analogue of the McCoy–Wu model. The special probability distribution of the magnetic field $h$ results in the splitting of the lattice into strips of finite width with definite boundary conditions. Since the free energy of such strips can be calculated exactly, the quenched free energy per site $F(p, T)$ and the specific heat $C(p, T)$ of our model can also be written as (Grinstein and Mukamel 1983)

$$-\beta F(p, T) = \tfrac{1}{2} \sum_{N=1}^{\infty} W_N(p)[f_+(N) + f_-(N)] \tag{1.2}$$

where $W_N(p) = p^2(1-p)^{N-1}$ is the probability of having a strip of width $(N+1)$. $f_+$ and $f_-$ denote the free energy densities (in the $M \to \infty$ limit) of these strips, corresponding to different directions of $h$ in the first and $(N+1)$th rows. In the case of $f_+$, spins in the two boundary rows point up (or down), whereas in the case of $f_-$, spins in the first row all point up, spins in the $(N+1)$th row all point down (or vice versa). Note that the boundary conditions for $f_+$, $f_-$ are more restricted that just periodic and antiperiodic.

Using the exact expression for $F(p, T)$ the existence of a Griffiths-like singularity at $T_c$ will be established for $p > 0$.

The paper is organised as follows. In § 2 the calculation of $F(p, T)$ is presented; in § 3 we calculate $C(p, T)$ and analyse its behaviour; § 4 contains the mathematical proof of the Griffiths singularity. In § 5, we summarise our results and make some remarks on the possible extension of the model to higher dimensions, on the question of the lower critical dimensionality and on the relevance to experimental systems. Some details of the calculations are presented in the appendices.

## 2. Calculation of the quenched free energy

The specific distribution of the magnetic field allows one to write the quenched free energy equation (1.2) as an average over free energy densities of finite strips. A non-zero magnetic field only acts on the first and last row, giving rise to the boundary conditions described in § 1. Partition functions of strips with finite magnetic fields of equal sign imposed on the boundary rows have been calculated by Au-Yang and Fisher (1980) using the Pfaffian technique. As we shall see, however, the transfer matrix technique (Schultz *et al* 1964) is more appropriate in our case.

In deriving (1.2) we have already used the spin-flip symmetry of the partition function for $(N+1)\times M$ strips

$$Z_{++}(M, N)=Z_{--}(M, N) \qquad Z_{+-}(M, N)=Z_{-+}(M, N) \tag{2.1}$$

where subscripts indicate the boundary conditions in obvious notation. Using this rather trivial property we show in appendix 1 that

$$\begin{aligned} Z_{++}(M, N)&=\tfrac{1}{2}\,\mathrm{Tr}(T_s^N P)\\ Z_{+-}(M, N)&=\tfrac{1}{2}\,\mathrm{Tr}(T_s^N PQ). \end{aligned} \tag{2.2}$$

Here $T_s$ is the symmetrised transfer matrix of the 2D IM (without magnetic field) (Schultz *et al* 1964, Hoever *et al* 1981),

$$P=\tfrac{1}{2}(|+\rangle\langle+|+|-\rangle\langle-|) \tag{2.3}$$

$$Q|\nu\rangle=|-\nu\rangle. \tag{2.4}$$

$|\nu\rangle$ describes an arbitrary spin state of a row; $|-\nu\rangle$ is the corresponding state with all spins flipped. $|+\rangle$ ($|-\rangle$) denotes the state with all spins up (down).

The advantage in using operators $P$, $Q$ is that they have simple representations in terms of Pauli spin matrices. Performing a Jordan–Wigner transformation to fermions and Fourier transforming we finally obtain (see appendix 1)

$$Z_{+(\pm)}(M, N)=\tfrac{1}{2}[Z_E(M, N)\pm Z_0(M, N)] \tag{2.5}$$

where

$$Z_E(M, N)=A_N(K)\prod_q \kappa_N(q) \qquad q=(\pi/M)k \qquad k=1, 3, \ldots, M-1 \tag{2.6}$$

$$Z_0(M, N)=A_N(K)\,\mathrm{e}^{2NK}\prod_q \kappa_N(q) \qquad q=(\pi/M)l \qquad l=2, 4, \ldots, M-2 \tag{2.7}$$

$$\kappa_N(q)=\cosh NE(q)+N^{-1}(C+C^{(*)}\cos q)\partial(\cosh NE(q))/\partial(\cosh E(q)) \tag{2.8}$$

$$\cosh E(q)=CC^{(*)}+\cos q \qquad E(q)\geqslant 0. \tag{2.9}$$

Here $C^{(*)}=\cosh 2K^{(*)}$, $S^{(*)}=\sinh 2K^{(*)}$ where $K^{(*)}$ is the dual coupling, $\exp(-2K^{(*)})=\tanh K$ and $A_N(K)=(2S)^{MN/2}$. $M$ is assumed to be even.

Using this in (1.2), we obtain

$$-\beta F(p, T)=\tfrac{1}{2}\sum_{N=1}^{\infty} W_N(p)\lim_{M\to\infty}\frac{1}{M}[2\ln Z_E(M, N)+\ln(1-\rho(M, N)^2)] \tag{2.10}$$

where

$$\rho(M, N)=Z_0(M, N)/Z_E(M, N). \tag{2.11}$$

In the first term within the brackets the $M \to \infty$ limit is performed by replacing the $q$ sum by an integral:

$$\frac{1}{M} \ln Z_E(M, N) = N \ln(2S) + \int_0^\pi \frac{dq}{2\pi} \ln \kappa_N(q) \qquad M \to \infty. \tag{2.12}$$

The second term in (2.10) is harder to analyse. For large $M$ the quantities $Z_0$ and $Z_E$ are exponentially close to each other. To leading order we obtain (see appendix 2)

$$\rho(M, N) = 1 - 2z(N)^M \qquad M \gg 1 \tag{2.13}$$

where $z(N)$ is the zero of $\kappa_N(q)$ in equation (2.8) in the complex $z$ plane, $z = e^{iq}$ which is inside and closest to the unit circle.

The calculation of the zeros of $\kappa_N(q)$ is closely related to the distribution of wavenumbers in the IM transfer matrix with free edges (Abraham 1971) or its dual, the hard-edged case (Abraham and Martin-Löf 1973). In fact, it can be shown that by choosing the horizontal direction as the direction of transfer, i.e. transferring from column to column instead of from row to row as in (2.2), one encounters precisely this problem.

From (2.11)-(2.13) we get the final result for the quenched free energy

$$-\beta F(p, T) = \tfrac{1}{2} \ln(2S) + \tfrac{1}{2} \sum_{N=1}^{\infty} W_N(p) \left( \int_0^\pi \frac{dq}{\pi} \ln \kappa_N(q) + \ln|z(N)| \right). \tag{2.14}$$

Equation (2.14) is the starting point of our investigation of the critical properties of the RFIM, presented in the following sections.

We finish this section with some remarks on related models. From the representation (A1.18) of the projector $P$ it is obvious that instead of imposing infinite magnetic fields on the rows one may likewise use infinite horizontal couplings to align the spins within a given row. Thus our RFIM can also be viewed as a particular, as yet unsolved, realisation of the McCoy-Wu model (McCoy and Wu 1968, 1973) where in each row the horizontal couplings are $\infty$ or $K$ with probabilities $p$ or $1-p$, respectively.

If, on the other hand, only magnetic fields of equal sign are allowed, i.e. $h_{ij} = \infty$ or 0 with probabilities $p$ or $1-p$, then instead of the second term $\ln[1-\rho^2(M, N)]$ in (2.10) we obtain $\ln[1+\rho(M, N)]$ which because of (2.13) leads to a vanishing contribution as $M \to \infty$. The corresponding quenched free energy is then given by (2.14) with $z(N) \equiv 1$. This also clarifies the meaning of the two terms in (2.14). The first term describes the contribution from the global properties of the strips while the second gives the correction originating from the interfaces created by magnetic fields of different sign. One expects that the critical behaviour of our model is mainly determined by the global properties of the strips, i.e. by the first term in (2.14). In fact, we shall show in § 3 (see also appendix 2) that the correction term in (2.14) is irrelevant, i.e. it gives no contribution to the critical behaviour.

## 3. Critical behaviour

In § 2 we have obtained explicit expressions for the quenched free energy. Each term in the sum (2.14) is an analytic function of temperature. In the zero concentration limit, $p \to 0$, or equivalently, no magnetic field, we should get back the free energy of the 2D IM with a logarithmically divergent specific heat (Onsager 1944). Two questions then arise: how is this Onsager singularity restored in the limit $p \to 0$, and does the model (1.1) still show a phase transition for finite $p$?

To obtain a finite limit as $p \to 0$ in (2.14) the overall factor $p^2$ in $W_N(p)$ has to be compensated. This only happens if we can extract a term proportional to $N$ from the terms within the brackets since

$$\sum_{N=1}^{\infty} N W_N(p) = 1. \tag{3.1}$$

It suffices to investigate the first term in (2.14). Rewriting $\ln \kappa_N(q)$ as

$$\begin{aligned} &\ln\cosh NE(q) + \ln[1 + \tanh NE(q)\cos a(q)] \\ &\quad = NE(q) + \ln\tfrac{1}{2}[1 - \mathrm{e}^{-2NE(q)}] + \ln[1 + \tanh NE(q)\cos a(q)] \end{aligned} \tag{3.2}$$

it is straightforward to show that only the first term in (3.2) contributes to the Onsager result

$$-\beta F(p=0, T) = \tfrac{1}{2}\ln(2S) + \int_0^\pi \frac{\mathrm{d}q}{2\pi} E(q) \tag{3.3}$$

while the other two remaining terms vanish at $p = 0$.

The specific heat $C_0$ of the pure 2D IM diverges logarithmically at $T_c$ which is obtained by differentiating (3.3) twice with respect to $T$ and using (2.9). This divergence of $C_0$ at $T_c$ must show up in $C(p\to 0, T_c)$. Differentiating (3.2) twice with respect to $T$ and putting $T = T_c$ one finds after a lengthy, but straightforward, calculation that only the $\ln\cosh NE(q)$ term in (3.2) leads to a non-vanishing contribution at $T_c$ and $p \to 0$ given by

$$C(p\to 0, T_c) \sim -\sum_{N=1}^{\infty} W_N(p) N \int_0^\pi \frac{\mathrm{d}q}{2\pi} \frac{\tanh NE(q)}{\sinh E(q)}\bigg|_{T_c} \tag{3.4}$$

where at $T_c$

$$\cosh E(q) = 2 - \cos q. \tag{3.5}$$

It is the large $N$ part of the sum (3.4) which gives the important contribution as $p \to 0$. To evaluate the integral in (3.4) for large $N$, we split the range of integration into two parts, $\int_0^\pi = \int_0^{1/N} + \int_{1/N}^\pi$. The first integral is bounded as $N \to \infty$ and thus contributes only a constant to the specific heat. The second integral diverges as $N \to \infty$. To leading order we get

$$\int_{1/N}^\pi \mathrm{d}q \frac{\tanh NE(q)}{\sinh E(q)}\bigg|_{T_c} \sim \int_{1/N}^\pi \frac{\mathrm{d}q}{q} \sim \ln N \qquad N \gg 1 \tag{3.6}$$

where $E(q) \sim q$, $q \ll 1$, has been used. The singular part of the specific heat at $T_c$ is thus given by

$$C(p\to 0, T_c) \sim \sum_{N=1}^{\infty} W_N(p) N \ln N. \tag{3.7}$$

Rewriting the sum as an integral, this leads to

$$C(p\to 0, T_c) \sim -[p/\ln(1-p)]^2 \ln|\ln(1-p)| \sim -\ln(p) \qquad p \ll 1. \tag{3.8}$$

Consequently, when approaching the Onsager critical point ($p = 0$, $T = T_c$) along the $p$ axis or the $T$ axis the singularity of the specific heat is always logarithmic.

## 4. Weak singularity for $p > 0$

As shown in § 3 the specific heat at the critical temperature $T_c$ diverges logarithmically as $p \to 0$. Furthermore, the quenched free energy $F(p, T)$ converges to the free energy of the pure 2D IM $F(0, T)$. In particular, $\lim_{p\to 0} F(p, T)$ yields the correct logarithmic singularity at $T_c$.

In this section we prove that even for $p > 0$ the free energy $F(p, T)$ as a function of the temperature is non-analytic at $T_c$, although the singularity is very mild: every derivative of $F(p, T)$ with respect to $T$ exists at $T_c$. The Taylor expansion around $T_c$, however, is not summable since the expansion coefficients increase faster than exponentially. Singular behaviour of thermodynamic functions of IM with random fields has been discussed by Schwartz *et al* (1984).

From both physical and mathematical points of view the situation is analogous to that of the Griffiths singularity (Griffiths 1969). This singularity occurs in randomly diluted Ising ferromagnets where the quenched free energy is a weighted sum of free energies of finite clusters plus, above the percolation threshold, the contribution from the infinite cluster. If this infinite cluster is absent, i.e. below the percolation threshold, there is thus no spontaneous magnetisation for any $T > 0$. Still, below the critical temperature $T_c$ of the pure model the quenched free energy as a function of the external magnetic field $h$ is non-analytic at $h = 0$ in the whole concentration range. The singularity is caused by the zeros of the cluster partition function $Z(\zeta)$, $\zeta = e^{-\beta h}$, which, for $T < T_c$, accumulate to $\zeta = 1$ in the complex $\zeta$ plane as the cluster size increases. The mathematical proof exploits the Lee-Yang circle theorem (Yang and Lee 1952) claiming that all these zeros lie on the unit circle $|\zeta| = 1$. It was already conjectured by Griffiths (1969) and since then it has been generally expected (see e.g. Parisi 1982) that similar singularities in the temperature variable also occur in random models whose pure analogue undergoes a phase transition. However, no proof has been given because of the lack of sufficient information about the distribution of the zeros.

In our case the free energy is a weighted sum of free energies $f_N$ of strips of finite width $N$. Due to the one-dimensional character of these strips their free energy is an analytic function for any real, positive $T$. However, we shall show below that $f_N(T)$ has branch-cut singularities in the complex $T$ plane which for $N \to \infty$ accumulate at $T_c$ and at no other real positive $T$. Moreover, we prove an approximate Lee-Yang circle theorem in the variable $S \equiv \sinh 2K$: the singularities of $f_N$ as a function of $S$ lie in the $1/N$ neighbourhood of the unit circle $|S| = 1$.

In order to see these properties we perform the $q$ integration in the free energy (2.14). This is easily done by observing that $\kappa_N(q)$ is an $N$th order polynomial in $\cos q$ and thus can be written as

$$\kappa_N(q) = 2^{N-1}(C^{(*)} + 1) \prod_{n=1}^{N} [\cos q - Q_{N,n}(S)] \tag{4.1}$$

with still unknown zeros $Q_{N,n}(S)$. Using (4.1) in (2.14) we find, apart from uninteresting terms,

$$-\beta F = \tfrac{1}{2} \sum_{N=1}^{\infty} W_N(p)\left( \sum_{n=1}^{N} \ln[-Q_{N,n}(s) + (Q_{N,n}(S)^2 - 1)^{1/2}] + \ln|z(N)| \right). \tag{4.2}$$

In the following the second term within the brackets in (4.2) will be neglected. This term cancels exactly against one term of the $n$ sum (see appendix 2), thereby producing only notational changes in the following proof.

We have not been able to derive exact expressions for the zeros $Q_{N,n}(S)$ for general $N$. Fortunately, they are not needed in our analysis. An approximate solution of $\kappa_N = 0$ relies on the observation that the zeros of the $\cosh NE(q)$ term in (2.8) are easy to obtain and the corrections coming from the second term in (2.8) are small for large $N$. Indeed, $\cosh NE(q)=0$ implies

$$\cosh E(q) \equiv S+S^{-1}+\cos q=\cos \alpha_{N,n} \qquad n=1,2,\ldots,N \tag{4.3}$$

where $\alpha_{N,n}=(2n-1)\pi/(2N)$. The solution of (4.3) is

$$Q^{(0)}_{N,n}(S)=\cos \alpha_{N,n}-S-S^{-1}. \tag{4.4}$$

Linearising (4.1) around $Q^{(0)}_{N,n}(S)$ and solving the resulting equation yields

$$Q_{N,n}(S)=Q^{(p)}_{N,n}(S)-(C^{(*)}/N)(S^{(*)}-\cos \alpha_{N,n}). \tag{4.5}$$

This expression is used to establish the location of the branch points of $[Q^2_{N,n}(S)-1]^{1/2}$ in the complex $S$ plane. For the physically allowed real positive values of $S$, this quantity cannot vanish unless $S=1$ and $N\to\infty$, $n/N\to 0$. The interesting branch points are therefore the solutions of

$$Q_{N,n}(S)=-1. \tag{4.6}$$

Replacing $Q$ by $Q^{(0)}$ and using (4.4) we find two solutions $S_\pm$ both lying on the unit circle

$$S_\pm=\exp(\pm\mathrm{i}\phi) \qquad \cos\phi=\cos^2(\tfrac{1}{2}\alpha_{N,n}) \qquad 0\leqslant\phi\leqslant\tfrac{1}{2}\pi. \tag{4.7}$$

Looking for a solution of (4.6) in the form $\hat{S}=S_0+\varepsilon$, $S_0=S_\pm$ then yields

$$\varepsilon=N^{-1}(1+S_0^2)^{1/2}(1-S_0\cos\alpha_{N,n}). \tag{4.8}$$

Therefore $\varepsilon$ is at most of order $1/N$ for any $n$ and $N$ and, in particular, if $\hat{S}$ is close to 1, i.e. $n/N\ll 1$, then from (4.8) $\varepsilon\sim 1/N^2$. Hence we establish that with $N\to\infty$, the branch points accumulate to the unit circle and, in particular, to $S=1$ if $n/N\to 0$ also holds. This gives rise to a non-analyticity in the total free energy at $S=1$, i.e. at $T=T_c$, which is easier to see after differentiation with respect to $S$:

$$\frac{\partial}{\partial S}(-\beta F)=\sum_{N=1}^{\infty}\sum_{n=1}^{N}\eta_{N,n}(S)[Q_{N,n}(S)+1]^{-1/2}. \tag{4.9}$$

It is not difficult to show that $\Sigma|\eta_{N,n}(S)|$ is finite in the neighbourhood of $S=1$. Furthermore, in whatever small neighbourhood of $S=1$, one can find an $\hat{S}=\hat{S}(N,n)$ defined by (4.7) and (4.8). This $\hat{S}$ is an isolated singularity, since it is separated from the unit circle, i.e. from the location of the possible accumulation points. Taking then $S=r\hat{S}$, the sum (4.9) diverges as $r\to 1$. It therefore cannot be analytic as $S=1$.

The nature of this singularity can be revealed by inspecting (4.2). For any given $N$ the sum of $n$, i.e. the free energy $f_N$ of a finite strip, is an analytic function of $S$ at $S=1$. However, the radius of convergence of its Taylor expansion around $S=1$ decreases as $1/N$, being equal to the distance of the set of branch points from 1. Therefore, the coefficient $a_m(N)$ in the expansion $f_N=\Sigma_{m=0}^{\infty}a_m(N)(S-1)^m$ must increase like $N^m$. Now the formal expansion of the total free energy $F$ in powers of $S-1$ still exists for $p>0$, but the coefficient of $(S-1)^m$ behaves asymptotically $(m\to\infty)$ as

$$a_m\sim m!\,[(1-p)/p]^m. \tag{4.10}$$

Therefore the series cannot be convergent for any $S$.

## 5. Summary and concluding remarks

In conclusion, we have solved exactly a 2D IM in RMF. We have given explicit formulae for the quenched free energy density and for the singular part of the specific heat. The model does not exhibit ordering for finite $p$. The singularity of the pure 2D IM at the Onsager critical point ($T_c$) manifests itself in the singular behaviour of the specific heat, namely $C(p, T_c) \sim_{p\to 0} -\ln p$. It has been shown that for positive $p$ the free energy $F(p, T)$ is not an analytic function of $T$ at $T = T_c$; it has a Griffiths singularity. We established the existence of all temperature derivatives of the free energy at $T = T_c$; the non-analyticity can therefore be interpreted as a phase transition of infinite order.

As pointed out earlier, for $M$ (the number of columns) = 1 our model reduces to that of Grinstein and Mukamel (1983). These authors argued that the specific distribution chosen for the magnetic field is not as unphysical as it might seem. We can also argue that the generalisation of our model to $d = 3$ ($h$ is the same within a plane) can probably have experimental relevance. As shown recently by Kirenzenow (1984) intercalation compounds may reveal disordered staging, i.e. the space between the planes of the host material is filled by the intercalant in a random sequence. If one can find an intercalant and host material with net magnetic moments and similar exchange interactions between these moments, then if $g_i \gg g_h$ ($g_i$ and $g_h$ refer to the Landé factors of the intercalant and host material, respectively) upon application of a strong external magnetic field in the direction perpendicular to the intercalant planes the situation $h \to \infty$ (or $-\infty$) and 0 can be, in principle, realised to a good approximation.

Our calculation cannot resolve the question of lower critical dimensionality in the case of uncorrelated randomness. However, the lower critical dimensionality of our model, generalised to arbitrary dimensionality can be established. The generalisation of the model means that we allow for $(d-1)$-dimensional correlation of the random magnetic field ($\varepsilon_d = d - 1$). It is easy to see that in $d = 3$ the free energy has a 'true' singularity (not just Griffiths-like) at the Onsager point of the 2D IM (and probably at other temperatures between the 2D and 3D critical temperatures as well) even for finite $p$. In the expression for $F(p, T)$, given by (1.2) (which is valid for any $d$), $f_-(2)$ (for example) carries this singularity. $f_-(2)$ is the free energy density of a system with three planes of Ising spins, with infinite and antiparallel magnetic fields on the first and third plane. Therefore, the effective magnetic field in the middle plane is zero and its statistical mechanics is given by the 2D pure IM. In conclusion, the lower critical dimensionality of our model is two.

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## Appendix 1

In this appendix we shall calculate the partition functions (2.1) using standard transfer matrix techniques (Schultz *et al* 1964). The partition function $Z_{++}(M, N)$ of an $(N+1)\times M$ strip with all spins pointing up in the first and $(N+1)$th row can be written as

$$Z_{++}(M, N) = \langle + | T^N | + \rangle \tag{A1.1}$$

where $T$ is the transfer matrix of the 2D IM. With a complete set of states $|\nu\rangle$ (A1.1) is rewritten as

$$Z_{++}(M,N)=\sum_{\nu}\langle\nu|T^N|+\rangle\langle+|\nu\rangle=\tfrac{1}{2}\operatorname{Tr}(T^NP) \tag{A1.2}$$

where (2.1) has been used and $P$ is the projector defined in (2.3). Similarly, we obtain

$$Z_{+-}(M,N)=\langle+|T^N|-\rangle=\tfrac{1}{2}\operatorname{Tr}(T^NQP) \tag{A1.3}$$

where $Q$ is the spin-flip operator (2.4).

The traces in (A1.2) and (A1.3) are appropriately performed using the eigenstates of the transfer operator $T$ which in terms of Pauli spin matrices $\sigma^i$, $i=1,2,3$ is given by (Schultz *et al* 1964, Hoever *et al* 1981)

$$T=A_N(K)T_1T_2. \tag{A1.4}$$

Here

$$T_1=\exp\left(K\sum_{j=1}^{M}\sigma_j^1\right) \tag{A1.5}$$

transfers from row to row and

$$T_2=\exp\left(K\sum_{j=1}^{N}\sigma_j^3\sigma_{j+1}^3\right) \tag{A1.6}$$

measures the interaction within a row. In (A1.6) periodic boundary conditions in the horizontal direction are chosen, i.e. $\sigma_{M+1}^3=\sigma_1^3$.

Although widely known, the results of the eigenvalue problem for the symmetrised operator $T_S=T_2^{1/2}T_1T_2^{1/2}$ will be summarised below, since the notation is subsequently needed.

The transfer operator $T_S$ acts in the $2^N$-dimensional space $S$ of spin states $\{|\nu\rangle\}$. Introducing fermion operators $C$, $C^+$ by a Jordan–Wigner transformation (Hoever *et al* 1981)

$$\sigma_j^1=2C_j^+C_j-1 \qquad -\sigma_j^3=(-1)^{\Sigma_{l<j}C_l^+C_l}(C_j^++C_j) \tag{A1.7}$$

we obtain

$$\sum_{j=1}^{M}\sigma_j^3\sigma_{j+1}^3=\sum_{j=1}^{M-1}(C_j^+-C_j)(C_{j+1}^++C_{j+1})-(-1)^{\mathcal{N}}(C_M^+-C_M)(C_{M+1}^++C_{M+1}) \tag{A1.8}$$

where $\mathcal{N}=\Sigma_{j=1}^{M}C_j^+C_j$ is the number of fermions. Obviously, $\mathcal{N}$ commutes with $T_S$. Therefore $T_S$ consists of two parts: $T_E$ acts in the even subspace $S_E$, $(-1)^{\mathcal{N}}=1$, and $T_0$ acts in the odd subspace $S_0$, $(-1)^{\mathcal{N}}=-1$. The periodic boundary conditions in horizontal direction then imply $C_{M+1}^{(+)}=(-1)^{\mathcal{N}+1}C_1^{(+)}$, such that within each subspace

$$\sum_{j=1}^{M}\sigma_j^3\sigma_{j+1}^3=\sum_{j=1}^{M}(C_j^+-C_j)(C_{j+1}^++C_{j+1}). \tag{A1.9}$$

Fourier transforming $C_j=M^{-1/2}\sum_q e^{iqj}C_q$, we obtain

$$T_S=\prod_{0<q<\pi}T(q)\begin{cases}1 & S_E\\ T(0)T(\pi) & S_0\end{cases} \tag{A1.10}$$

where

$$T(q)=u(q)\exp[E(q)\tau_q^3]u(q)^{-1} \qquad u(q)=\exp[-\tfrac{1}{2}(a_q+q)\tau_q^2] \tag{A1.11}$$

$$T(0)T(\pi)=\exp[2K(C_0^+C_0-C_\pi^+C_\pi)+2K^{(*)}(C_0^+C_0-C_\pi C_\pi^+)] \tag{A1.12}$$

$$\tau_q^3=C_q^+C_q+C_{-q}C_{-q}^+-1 \qquad \tau_q^2=C_q^+C_{-q}^+-C_qC_{-q} \qquad \tau_q^1\equiv \mathrm{i}\tau_q^3\tau_q^2 \tag{A1.13}$$

$$\cosh E(q)=CC^{(*)}+\cos q \qquad \sinh E(q)\,\mathrm{e}^{\mathrm{i}a_q}=C+C^{(*)}\cos q-\mathrm{i}S^{(*)}\sin q. \tag{A1.14}$$

The Fourier momenta in (A1.10) take the values

$$q_{\mathrm{E}}=(\pi/M)k \qquad k=1,3,\ldots,M-1 \qquad \text{even subspace } S_{\mathrm{E}} \tag{A1.15}$$

$$q_0=(\pi/M)l \qquad l=2,4,\ldots,M-2 \qquad \text{odd subspace } S_0. \tag{A1.16}$$

Note that $M$ is assumed to be even.

For each $q$ value in (A1.15) and (A1.16) we encounter four different fermion states: two singly occupied states with either the $q$ or the $(-q)$ fermion present and the states of zero or double occupancy. However, as shown below, only the space of the latter states, in which the $\tau$ operators (A1.13) have the properties of Pauli spin matrices, is needed in the traces (A1.2) and (A1.3).

Using the above formulae, the representation of the projection operator $P$ (2.3) is easily derived by observing that

$$P=\lim_{x\to\infty}\tfrac{1}{2}\exp\left(x\sum_{j=1}^{M}(\sigma_j^3\sigma_{j+1}^3-1)\right) \tag{A1.17}$$

which in terms of fermion operators is given by

$$P=\lim_{x\to\infty}\tfrac{1}{2}\prod_{0<q<\pi}\exp[2x(\tau_q^3\cos q+\tau_q^1\sin q-1)]\begin{cases}1 & S_{\mathrm{E}}\\ \exp[2x(C_0^+C_0-C_\pi C_\pi^+)] & S_0\end{cases}. \tag{A1.18}$$

From (A1.18) it is obvious that $P$ is non-zero only in the space of zero and double occupancy of the fermions. Restricting ourselves to this space and performing the limit we get

$$P=\tfrac{1}{2}\prod_{0<q<\pi}\tfrac{1}{2}(1+\tau_q^3\cos q+\tau_q^1\sin q)\begin{cases}1 & S_{\mathrm{E}}\\ C_0^+C_0C_\pi C_\pi^+ & S_0\end{cases}. \tag{A1.19}$$

For the spin-flip operator $Q$ (2.4) the Jordan–Wigner transformation (A1.7) simply leads to

$$Q=\prod_{j=1}^{M}\sigma_j^1=(-1)^{\mathcal{N}}=\begin{cases}1 & S_{\mathrm{E}}\\ -1 & S_0.\end{cases} \tag{A1.20}$$

Having derived all necessary formulae, we can perform the calculation of the partition functions (A1.2) and (A1.3) in a straightforward way. Separating the traces of the even and odd subspace we get using (A1.20)

$$Z_{+(\pm)}(M,N)=\tfrac{1}{2}[\mathrm{Tr}_{(S_{\mathrm{E}})}(T_S^NP)\pm\mathrm{Tr}_{(S_0)}(T^NP)] \tag{A1.21}$$

which is equation (2.5) in the text. The remaining traces are readily done using the representations (A1.11) and (A1.19). For $q\neq 0$, $\pi$ we obtain

$$\mathrm{Tr}\,T(q)^NP(q)=\tfrac{1}{2}[\mathrm{e}^{NE(q)}(1+\cos a_q)+\mathrm{e}^{-NE(q)}(1-\cos a_q)]=\kappa_N(q) \tag{A1.22}$$

yielding (2.6) for the even subspace $S_{\mathrm{E}}$, whereas in the odd subspace $S_0$ a factor $\mathrm{e}^{2NK}$ has to be added due to the $q=0$, $\pi$ terms in (A1.11) and (A1.19).

## Appendix 2

In this appendix we shall investigate the second term in (2.10) which originates from the two different boundary conditions for the strips. In addition we derive an expression, equivalent to (2.14), for the quenched free energy. This expression is then used to show that the second term in (2.10) is irrelevant for the critical properties of our model.

The last term in (2.10) is determined by the ratio $\rho(M, N)$ (2.11) which can be rewritten as

$$\rho(M, N) = \prod_{l=(-M/2)+1}^{M/2} \left[\kappa_N\left(\frac{2\pi}{M} l\right)\kappa_N\left(\frac{2\pi}{M}(l-\tfrac{1}{2})\right)^{-1}\right]^{1/2} \tag{A2.1}$$

where (2.6)-(2.8) have been used. As $\kappa_N(q)$ depends only on $\cos q$ we may expand $\ln \kappa_N(q)$ in a Fourier series

$$\ln \kappa_N(q) = \tfrac{1}{2}a_0(N) + \sum_{k=1}^{\infty} a_k(N)\cos(kq) \tag{A2.2}$$

with Fourier coefficients

$$a_k(N) = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}q \ln \kappa_N(q)\cos(kq). \tag{A2.3}$$

This yields together with (A2.1)

$$\ln \rho(M, N) = M \sum_{k=0}^{\infty} a_{(2k+1)M}(N). \tag{A2.4}$$

On the other hand, $\kappa_N(q)$ (2.8) is an $N$th order polynomial in $\cos q$ and can therefore be written as

$$\kappa_N(q) = 2^{N-1}(C^{(*)}+1)\prod_{n=1}^{N}(\cos q - Q_{N,n}). \tag{A2.5}$$

Below we shall show that $Q_{N,n} < -1$, $n = 1, \ldots, N$. Defining $-2Q_{N,n} \equiv z_{N,n} + z_{N,n}^{-1}$, $0 \leqslant z_{N,n} < 1$, we get from (A2.3)

$$a_k(N) = -\frac{2}{M}\sum_{n=1}^{N}(-z_{N,n})^k \qquad k \geqslant 1 \tag{A2.6}$$

such that

$$\ln \rho(M, N) = -2\sum_{n=1}^{N} z_{N,n}^{M}\left(1 - z_{N,m}^{2M}\right)^{-1} \tag{A2.7}$$

i.e. $\rho(M, N)$ is determined by the 'zeros' $z_{N,n}$, $n = 1, \ldots, N$ of $\kappa_N(q)$.

For sufficiently large $M$ only the largest term, determined by the largest zero $z_{N,n_0} \equiv z(N)$, contributes to the sum in (A2.7), i.e. to leading order

$$\ln \rho(M, N) \simeq -2z(N)^M \qquad M \gg 1 \tag{A2.8}$$

such that

$$\rho(M, N) \simeq 1 - 2z(N)^M \qquad M \gg 1 \tag{A2.9}$$

which is (2.13) in the text.

The quenched free energy (2.14) can also be expressed in terms of the zeros $z_{N,n}$. Using (A2.5) in (2.14) we obtain apart from uninteresting terms

$$-\beta F(p, T)=\tfrac{1}{2}\sum_{N=1}^{\infty} W_N(p)\left(-\sum_{n=1}^{N}\ln(-z_{N,n})+\ln(-z_{N,n_0})\right) \qquad \text{(A2.10)}$$

i.e. the $n=n_0$ term cancels exactly.

The above expression (A2.10) shows that the second term is of order $1/N$ as compared to the $n$ sum and may therefore be neglected in the considerations of § 3. Furthermore, the proof of the Griffiths singularity in § 4 is modified only in an unessential way if this second term is taken into account.

It remains to show that $Q_{N,n}<-1$, $n=1,\ldots,N$ for $T\geqslant 0$. Using (2.9) we rewrite (A2.5) as

$$\kappa_N(q)=2^{N-1}(C^{(*)}+1)\prod_{n=1}^{N}[\cosh E(q)-\tfrac{1}{2}(x_{N,n}+x_{N,n}^{-1})] \qquad \text{(A2.11)}$$

where

$$\tfrac{1}{2}(x_{N,n}+x_{N,n}^{-1})=Q_{N,n}+CC^{(*)} \qquad \text{(A2.12)}$$

Rearrangement of (A2.11) yields that the quantities $x_{N,n}$ are the solutions of

$$x^{2N}=\frac{1}{AB}\frac{(x-A)(x-B)}{(x-A^{-1})(x-B^{-1})} \qquad x=e^{E(q)} \qquad \text{(A2.13)}$$

where

$$A=e^{2(K+K^{(*)})} \qquad B=e^{2(K-K^{(*)})}. \qquad \text{(A2.14)}$$

Equation (A2.13) has been investigated by Abraham and Martin-Löf (1973) where it was shown that for $T\geqslant T_c(B\geqslant 1)$ all its solutions lie on the unit circle, while for $T>T_c(B<1)$ also two real solutions $x_0$, $x_0^{-1}$ appear. Since $CC^{(*)}\geqslant 2$ for all $T$, we immediately establish that all $Q_{n,N}\leqslant -1$ with only one possible exception for $T>T_c$. However, as $\kappa_N(\pi/2)>0$ this last one must also be negative. Furthermore, all $Q_{N,n}\neq -1$ since $\kappa_N(q)$ does not vanish for $0\leqslant q\leqslant 2\pi$ (cf (A1.22)).

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